ANALYTICAL SOLUTION: THE ONE-DIMENSIONAL MELTING OF METALS BY A CONCENTRATED ENERGY BEAM

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The authors study the problem of melting of a massive metal target with a constant surface heat source up to the melting temperature. Analytical expressions have been obtained to describe the temperature field and the melting depth to an accuracy of about one half percent.

A very important special feature of the action of a concentrated energy flux on a solid is the phase transition of the first kind. One meets melting of metals by an energy flux in a variety of scientific and technical applications. However, the experimental possibilities for investigating this process are significantly limited, and as a rule one can obtain information only on integral effects. In essence,the methods of mathematical modeling remain as yet the sole means of a most complete description of the dynamics of interaction of thermal radiation with matter. The solution of the melting problem has been addressed in a whole series of papers [1-7]. The authors of [1] used an ablation model which assumed that the melt was removed from the heat source action zone. In other papers, melting has been examined with the liquid phase present and being heated. In most cases one must determine the temperature fields and the metal melt depth as a function of the thermophysical properties of the metal and the parameters of the energy flux incident on the target. For this purpose the approximate analytical method of Biot was used in [5], and numerical modeling with a finite difference scheme was used in [7]. Numerical modeling with an explicity determined melt front requires a powerful computer or writing of a rather complex program. Defects of the Biot method are that one cannot determine its accuracy from the analysis itself, that it is not clear how to increase the accuracy is needed, and the fact that the complexity of the base is not correlated with the attainable accuracy (the error in determining the melt depth is up to 15%). Thus, none of the above methods is at once simple, accurate and nonlaborious. But these three conditions are satisfied, we think, by the solution algorithm for the above problem described in this paper.

A constant heat flux falls on a massive target and is absorbed on the surface. The metal is heated to its melting temperature, at which time a surface melt layer forms whose outer boundary we consider as fixed, while the inner boundary begins to move inside the material. At the moving melt front there is absorption of the latent heat of fusion. We consider the thermophysical properties of the metal and its melt to be constant and identical. We shall examine the heating up to temperatures close to the boiling temperature of the metal, and we consider the incident heat flux density to be so large that we may neglect heat loss from the target surface due to convection and radiation. Then the mathematical formulation of the problem is as follows:

$$a\frac{\partial^2 T_1}{\partial z^2}=\frac{\partial T_1}{\partial t},\ z\geqslant 0,\ 0\leqslant t\leqslant t_m,$$

$$-\lambda\frac{\partial T_1}{\partial z}\bigg|_{z=0}=q,\ T_1(t=0,z)=T_1(t_1,\ z=\infty)=T_0, \tag{1}$$

$$a\frac{\partial^2 T_2}{\partial z^2}=\frac{\partial T_2}{\partial t},\ 0\leqslant z\leqslant s(t),\ t\geqslant t_m,\ a\frac{\partial^2 T_1}{\partial z^2}=\frac{\partial T_1}{\partial t},\ s(t)\leqslant z,\ t\geqslant t_m,$$

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$$-\lambda\left.\frac{\partial T_2}{\partial z}\right|_{z=0}=q,\ s(t_m)=0,$$

$$-\lambda\left.\frac{\partial T_2}{\partial z}\right|_{z=s(t)}=-\lambda\left.\frac{\partial T_1}{\partial z}\right|_{z=s(t)}+\rho L\frac{ds(t)}{dt}, \tag{1}$$

$$T_1(t,\ z=\infty)=T_0,\ T_1(t,\ s(t))=T_2(t,\ s(t))=T_m.$$

Using the heat source function of [6], we can represent the temperature field after the start of target melting as follows:

$$T(t,\ z)=T_0+\frac{q}{c\rho\sqrt{\pi a}}\int_0^t\frac{dt'}{\sqrt{t-t'}}\exp\left(-\frac{z^2}{4a(t-t')}\right)-$$
$$-\int_{t_m}^t dt'\frac{Ls'(t')}{2c\sqrt{\pi a(t-t')}}\exp\left(-\frac{(z-s(t'))^2}{4a(t-t')}\right)-\int_{t_m}^t dt'\frac{Ls'(t')}{2c\sqrt{\pi a(t-t')}}\exp\left(-\frac{(z+s(t'))^2}{4a(t-t')}\right). \tag{2}$$

We introduce the dimensionless parameters of the process:

$$\tau=\frac{t}{t_m},\ \tau'=\frac{t'}{t_m},\ \xi(\tau)=\frac{s(t)}{\sqrt{at_m}},\ \xi(\tau')=\frac{s(t')}{\sqrt{at_m}},$$
$$T_m=\frac{2q\sqrt{t_m}}{c\rho\sqrt{\pi a}},\ B=\frac{L}{2\sqrt{\pi}cT_m},\ A=\frac{z}{s(t)},\ f(\tau,\ A)=\frac{T(t,\ z)-T_0}{T_m}.$$

The integrodifferential Eq. (2) has the as yet unknown function $\xi(\tau')$. We seek it by the method of successive approximations.

As a first approximation for $\xi(\tau')$ we take $\xi(\tau')=\xi(\tau)(\tau'-1)/(\tau-1)$. As subsequent calculations have shown, for metals up to the boiling temperature, the error of the first approximation for $\xi(\tau')$ is about 3.7%. After replacing the variables from Eq. (2) we obtain:

$$f(t,\ A)=\sqrt{\tau}\int_0^1 d\eta\exp\left(-\frac{\gamma^2}{4\eta^2}\right)-2B\alpha\int_0^1 d\eta\exp\left(-\frac{\alpha^2(1-A-\eta^2)^2}{4\eta^2}\right)-$$
$$-2B\alpha\int_0^1 d\eta\exp\left(-\frac{\alpha^2(1+A-\eta^2)^2}{4\eta^2}\right),\ \gamma=\frac{A\xi(\tau)}{\sqrt{\tau}},\ \alpha=\frac{\xi(\tau)}{\sqrt{\tau-1}}. \tag{3}$$

Computing approximately the first integral in Eq. (3), we have

$$\int_0^1 d\eta\exp\left(-\frac{\gamma^2}{4\eta^2}\right)=\exp\left(-\frac{\gamma^2}{4}\right)\left(1+\frac{a_1\gamma}{1+a_2\gamma}+\frac{\gamma^2}{2}\right)^{-1},$$
$$a_1=\frac{\sqrt{\pi}}{2},\ a_2=\frac{2}{\sqrt{\pi}}\left(1-\frac{\pi}{4}\right).$$

The error of this approximation, as numerical calculation on a computer has shown, does exceed 0.15% of the maximum value for $\gamma=0$, up to the value $\gamma=2$.

The second and third integrals in Eq. (3) reduce to the following:

$$I(\alpha,\ D)=\int_0^1 d\eta\exp\left(-\frac{\alpha^2(D-\eta^2)^2}{4\eta^2}\right).$$

For various values of the parameter D this integral is represented in the form

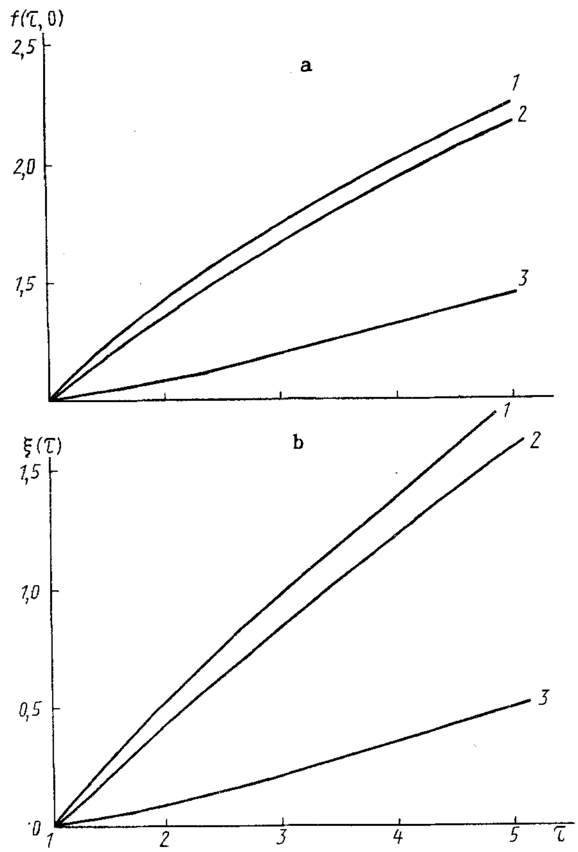

Fig. 1. Time dependence of the metal surface temperature (a) and of the melt depth (b) for different values of the parameter B: 1) B = 0; 2) 0.055; 3) 1.

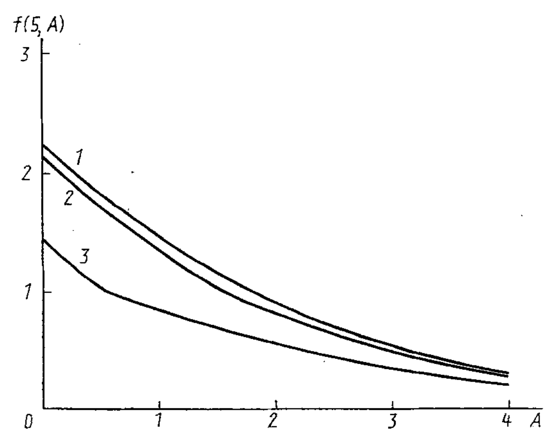

Fig. 2. Spatial dependence of the temperature at time $\tau = 5$ for various values of the parameter B: 1) B = 0; 2) 0.055; 3) 1.

$$D \geqslant 1:\ I(\alpha,\ D) = \exp\left(-\frac{\alpha^2 (D-1)}{4}\right)(1 + b_1\alpha + b_2\alpha^2)^{-1},$$

$$b_1 = \frac{\sqrt{\pi} D}{2},\ b_2 = \frac{D^2(\pi - 2)}{4} - \frac{1}{6},\ 0 \leqslant D \leqslant 1:\ I(\alpha,\ D) =$$

$$= \frac{1}{\sqrt{1 + c_1\alpha + c_2\alpha^2}},\ c_1 = \sqrt{\pi} D,\ c_2 = \frac{D^2(3\pi - 2)}{4} - D + \frac{1}{6},$$

$$-1 \leqslant D \leqslant 0:\ I(\alpha,\ D) = \exp(-\alpha^2 |D|)(1 + f_1\alpha + f_2\alpha^2)^{-0,5},$$

$$f_1 = \sqrt{\pi}|D|,\ f_2 = \frac{D^2(3\pi - 2)}{4} - |D| + \frac{1}{6},$$

$$D \leqslant -1:\ I(\alpha,\ D) = \exp\left(-\frac{\alpha^2(1 + |D|)}{4}\right)(1 + g_1\alpha + g_2\alpha^2)^{-1},$$

$$g_1 = \frac{\sqrt{\pi}|D|}{2},\ g_2 = \frac{D^2(\pi - 2)}{4} + \frac{1}{12}.$$

Calculations on a computer have shown that the error of these approximations up to the value $\alpha = 0.8$ is not more than 0.5% of the maximum values for $\alpha = 0$. And since the physical model used is limited above by the boiling temperature of the metals ($\sim 2T_m$), then τ does not exceed 5, and α does not exceed 0.8. In addition, the parameter B for metals is ~0.05, which reduces the error in computing $f(\tau, A)$ by an order of magnitude further.

To determine the time dependence of the metal melt depth we assume $A = 1$, $f(\tau, A) = 1$ and obtain the algebraic equation

$$1 = \sqrt{\tau}\exp\left(-\frac{\gamma^2}{4}\right)\left(1 + \frac{a_1\gamma}{1 + a_2\gamma} + \frac{\gamma^2}{2}\right)^{-1} -$$

$$-2B\alpha\exp\left(-\frac{\alpha^2}{4}\right)(1 + b_1\alpha + b_2\alpha^2)^{-1} - 2B\alpha(1 + c_1\alpha + c_2\alpha^2)^{-0,5}, \tag{4}$$

$$\gamma = \frac{\xi(\tau)}{\sqrt{\tau}},\quad \alpha = \frac{\xi(\tau)}{\sqrt{\tau - 1}}.$$

We solve the desired Eq. (4) by the method of successive approximations, taking account of the smallness of the parameters B, α and γ. At the third iteration, the error of determing $\xi(\tau)$ is not more than 0.07%. It remains to add that since the second approximation obtained by solving Eq. (4), differs from the first approximation $\xi(\tau')$ by not more than 3.7%, the difference between the second and third approximations (this being in fact an accurate solution) will not be more than 0.15%. This, the total error of determining $\xi(\tau)$ is about 0.5%, which is less than in the Biot method by a factor of 30.

Figures 1 and 2 show graphs of the timewise and spatial dependence of the temperature field $f(\tau, A)$ and the target melt depth $\xi(\tau)$ for various values of the parameter B.

Conclusion. We have developed a simple, accurate, and nonlaborious method for analytical solution of the one-dimensional problem of melting of a metal by a concentrated energy flux, up to temperatures near boiling.

NOTATION

t, τ, time of action of the thermal irradiation on the target; z, A spatial coordinate, reckoned into the interior of the target from its surface; T_1, T_2 temperature fields in the solid and liquid metal phases; T, f, general temperature fields in the metal; λ, thermal conductivity; a thermal diffusivity; q, heat flux density absorbed by the target; T_0, initial target temperature; T_m, melting temperature of the metal; t_m, heating time of the target surface to the melting temperature; s, ξ, melt depth of the metal; ρ, density; L, specific heat of fusion; c, specific heat.

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